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Noncommutative Geometry and the Standard Model

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Abstract

Connes' noncommutative approach to the standard model of electromagnetic, weak and strong forces is sketched as well as its unification with general relativity.

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1 Spectral triples

Noncommutative geometry [1, 2, 3] equips Riemannian spin manifolds with an uncertainty relation just as quantum mechanics equips phase space with Heisenberg's uncertainty relation. Major building blocks of the theory are therefore the algebra of observables and a representation on a Hilbert space. More precisely a **real, even spectral triple** is given by five items

- \mathcal{A} is a real, associative algebra with unit 1 and involution $*$. (Its elements can be called observables.)
- \mathcal{H} is a complex Hilbert space carrying a faithful representation ρ of the algebra. (Its elements are the wave functions.)
- \mathcal{D} is a selfadjoint operator on \mathcal{H} of compact resolvent. We call it a Dirac operator.
- J is an anti-unitary operator on \mathcal{H} . We call it real structure or charge conjugation.
- χ is a unitary operator on \mathcal{H} . We call it chirality.

We require the following axioms to hold:

- $J^2 = -1$ in four dimensions ($J^2 = 1$ in zero dimension).
- $[\rho(a), J\rho(\tilde{a})J^{-1}] = 0$ for all $a, \tilde{a} \in \mathcal{A}$.
- $\mathcal{D}J = J\mathcal{D}, \quad J\chi = \chi J, \quad \mathcal{D}\chi = -\chi\mathcal{D}$.
- $[\mathcal{D}, \rho(a)]$ is bounded for all $a \in \mathcal{A}$ and $[[\mathcal{D}, \rho(a)], J\rho(\tilde{a})J^{-1}] = 0$ for all $a, \tilde{a} \in \mathcal{A}$. This property is called first order condition because in the first example below, it states that the genuine Dirac operator is a first order differential operator.
- $\chi^2 = 1$ and $[\chi, \rho(a)] = 0$ for all $a \in \mathcal{A}$. These properties allow the decomposition $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$.
- We suppose the kernel of \mathcal{D} to contain no non-trivial subspace invariant under the representation of the algebra \mathcal{A} ("non-degeneracy").
- There are three more properties, that we do not spell out, orientability, which relates the chirality to a volume form, Poincaré duality and regularity, which states that the observables $a \in \mathcal{A}$ are differentiable.

The calibrating, commutative example is given by an even dimensional, compact, Riemannian spin manifold M (Euclidean signature), which for concreteness we take to be 4-dimensional (Euclidean timespace). Let $\mathcal{A} = \mathcal{C}^\infty(M)$, the algebra of complex valued differentiable functions, $\mathcal{H} = \mathcal{L}^2(\mathcal{S})$, the square integrable sections of the spinor bundle, $\mathcal{D} = \not{D}$, the genuine Dirac operator, $J = \gamma^0\gamma^2 \circ$ complex conjugation and $\chi = \gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$. The γ^a are the Dirac matrices.

Another commutative example, the two-point space, is discrete or 0-dimensional:

$$\mathcal{A} = \mathbb{C}_L \oplus \mathbb{C}_R \ni (a_L, a_R), \quad \mathcal{H} = \mathbb{C}^4, \quad \rho(a_L, a_R) = \begin{pmatrix} a_L & 0 & 0 & 0 \\ 0 & a_R & 0 & 0 \\ 0 & 0 & \bar{a}_R & 0 \\ 0 & 0 & 0 & \bar{a}_L \end{pmatrix}, \quad (1)$$

$$\mathcal{D} = \begin{pmatrix} 0 & m & 0 & 0 \\ \bar{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{m} \\ 0 & 0 & m & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \circ c \circ c, \quad \chi = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

As we will see, the parameter $m \in \mathbb{C}$ will determine the distance between the two points as $1/|m|$. For 0-dimensional triples the regularity axiom is empty, while orientability means that the chirality can be written as a finite sum,

$$\chi = \sum_j \rho(a_j) J \rho(\tilde{a}_j) J^{-1}, \quad a_j, \tilde{a}_j \in \mathcal{A}. \quad (3)$$

The Poincaré duality says that the intersection form

$$\cap_{ij} := \text{tr} [\chi \rho(p_i) J \rho(p_j) J^{-1}] \quad (4)$$

must be non-degenerate, where the p_j are a set of minimal projectors of \mathcal{A} . The two-point space is orientable, $\chi = \rho(-1, 1) J \rho(-1, 1) J^{-1}$. It also satisfies Poincaré duality, there are two minimal projectors, $p_1 = (1, 0)$, $p_2 = (0, 1)$, and the intersection form is $\cap = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$.

Connes' reconstruction theorem [3] states that all *commutative*, real (even) spectral triples are of the above type: they come from a (even or 0-dimensional) compact Riemannian spin manifold M . In this reconstruction the Dirac operator plays three roles: it allows to reconstruct the dimension, the metric and the differential structure:

The **dimension** is a local property of space. It can be retrieved from the asymptotic behaviour of the spectrum of the Dirac operator for large eigenvalues. Since M is compact, the spectrum is discrete. Let us order the eigenvalues, $\dots \lambda_{n-1} \leq \lambda_n \leq \lambda_{n+1} \dots$. Then Weyl's spectral theorem states that the eigenvalues grow asymptotically as $n^{1/\dim M}$. To explore a local property of space we only need the high energy part of the spectrum. This is in nice agreement with our intuition from quantum mechanics and motivates the name 'spectral triple'.

The **metric** can be reconstructed from the commutative spectral triple by Connes' distance formula (5) below. In the commutative case points $x \in M$ are reconstructed as pure states. The general definition of a pure state of course does not use the commutativity. A state δ of the algebra \mathcal{A} is a linear form on \mathcal{A} , that is normalized, $\delta(1) = 1$, and positive, $\delta(a^*a) \geq 0$ for all $a \in \mathcal{A}$. A state is pure if it cannot be written as a convex combination of two states. For the calibrating example, there is a one-to-one correspondence between points $x \in M$ and pure states δ_x defined by the Dirac distribution,

$\delta_x(a) := a(x) = \int_M \delta_x(y) a(y) d^4 y$. The geodesic distance between two points x and y is reconstructed from the triple as:

$$\sup \{ |\delta_x(a) - \delta_y(a)|; a \in \mathcal{C}^\infty(M) \text{ such that } ||[\not{D}, \rho(a)]|| \leq 1 \}. \quad (5)$$

Note that Connes' distance formula continues to make sense for non-connected manifolds, like discrete spaces.

Differential forms, for example of degree one like da for a function $a \in \mathcal{A}$, are reconstructed as $(-i)[\not{D}, \rho(a)]$. This is again motivated from quantum mechanics. Indeed in a 1+0 dimensional timespace da is just the time derivative of the observable a and is associated with the commutator of the Hamilton operator with a .

As in quantum mechanics, we define a noncommutative geometry by a real spectral triple with noncommutative algebra \mathcal{A} . Here is a 0-dimensional noncommutative example:

$$\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \ni (a, b, c), \quad \mathcal{H} = \mathbb{C}^{30}. \quad (6)$$

By \mathbb{H} we denote the quaternion algebra, complex 2×2 matrices of the form

$$a = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}. \text{ The representation is : } \rho(a, b, c) := \begin{pmatrix} \rho_L & 0 & 0 & 0 \\ 0 & \rho_R & 0 & 0 \\ 0 & 0 & \bar{\rho}_L^c & 0 \\ 0 & 0 & 0 & \bar{\rho}_R^c \end{pmatrix} \quad (7)$$

with

$$\rho_L(a) := \begin{pmatrix} a \otimes 1_3 & 0 \\ 0 & a \end{pmatrix}, \quad \rho_R(b) := \begin{pmatrix} b 1_3 & 0 & 0 \\ 0 & \bar{b} 1_3 & 0 \\ 0 & 0 & \bar{b} \end{pmatrix}, \quad (8)$$

$$\rho_L^c(b, c) := \begin{pmatrix} 1_2 \otimes c & 0 \\ 0 & \bar{b} 1_2 \end{pmatrix}, \quad \rho_R^c(b, c) := \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \bar{b} \end{pmatrix}, \quad (9)$$

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\mathcal{M}} \\ 0 & 0 & \bar{\mathcal{M}}^* & 0 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \otimes 1_3 & 0 \\ 0 & \begin{pmatrix} 0 \\ m_e \end{pmatrix} \end{pmatrix}, \quad (10)$$

with three positive numbers m_u , m_d and m_e ,

$$J = \begin{pmatrix} 0 & 1_{15} \\ 1_{15} & 0 \end{pmatrix} \circ \text{ complex conjugation}, \quad \chi = \begin{pmatrix} -1_8 & 0 & 0 & 0 \\ 0 & 1_7 & 0 & 0 \\ 0 & 0 & -1_8 & 0 \\ 0 & 0 & 0 & 1_7 \end{pmatrix}. \quad (11)$$

This triple is orientable, $\chi = \rho(-1_2, 1, 1_3)J\rho(-1_2, 1, 1_3)J^{-1}$, and Poincaré duality holds. We have three minimal projectors,

$$p_1 = (1_2, 0, 0), \quad p_2 = (0, 1, 0), \quad p_3 = \left(0, 0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \quad (12)$$

and the intersection form

$$\cap = -2 \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad (13)$$

is non-degenerate.

One of the attractive features of spectral triples is that they describe continuous and discrete spaces in the same language and therefore define differential forms also for discrete spaces. Another attractive feature is tensorisation. Given two real, even spectral triples, $(\mathcal{A}_j, \mathcal{H}_j, \dots)$, $j = 1, 2$, we get a third by taking $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\mathcal{D} = \mathcal{D}_1 \otimes 1_2 + \chi_1 \otimes \mathcal{D}_2$, $J = J_1 \otimes J_2$ and $\chi = \chi_1 \otimes \chi_2$. The other obvious choice for the Dirac operator, $\mathcal{D}_1 \otimes \chi_2 + 1_1 \otimes \mathcal{D}_2$, is unitarily equivalent to the first one. In the commutative case this tensor product simply describes the direct product of spaces, $\mathcal{C}^\infty(M_1) \otimes \mathcal{C}^\infty(M_2) = \mathcal{C}^\infty(M_1 \times M_2)$, $\mathcal{L}^2(\mathcal{S}(M_1)) \otimes \mathcal{L}^2(\mathcal{S}(M_2)) = \mathcal{L}^2(\mathcal{S}(M_1 \times M_2)), \dots$

We define a (4-dimensional) **almost commutative spectral triple** to be the tensor product of the commutative spectral triple coming from a (4-dimensional) space M and a noncommutative 0-dimensional spectral triple. An almost commutative space therefore has an infinite number of commutative degrees of freedom plus a finite number of noncommutative degrees of freedom. Motivation for almost commutative geometries comes from Pauli spinors where the Pauli matrices generate the quaternions or from Kaluza-Klein theories with the fifth dimension discrete and fuzzy [4]. Note that the genuine Dirac operator on M is massless. By the tensorisation with 0-dimensional triples, commutative or not, it becomes massive and \mathcal{M} is the fermionic mass matrix.

At present, we only have a few examples of truly noncommutative geometries, the first being the kinematics of quantum mechanics: in its compact version the noncommutative torus [5] or in its noncompact version the Moyal plane [6].

2 The spectral action

Every spectral triple carries a natural action, the spectral action [3, 7]. Its configuration space \mathcal{F} is an affine space constructed by extending Einstein's equivalence principle to noncommutative spaces:

$$\mathcal{F} := \left\{ \mathcal{D}_f = \sum_{\text{finite}} r_j L(\sigma_j) \mathcal{D} L(\sigma_j)^{-1}, \quad r_j \in \mathbb{R}, \quad \sigma_j \in \text{Aut}(\mathcal{A}), \right\}. \quad (14)$$

Indeed in the commutative case, every automorphism of $\mathcal{C}^\infty(M)$ is a diffeomorphism σ of M , a general coordinate transformation. $L(\sigma)$ is its double-valued lift [8] acting on

spinors $\psi \in \mathcal{L}^2(\mathcal{S})$, in a coordinate neighbourhood

$$(L(\sigma)\psi)(x) = (S(\Lambda(\sigma, g))|_{\sigma^{-1}(x)} \psi)(\sigma^{-1}(x)), \quad (15)$$

with

$$\begin{aligned} S : SO(n) &\longrightarrow Spin(n) & \Lambda(\sigma, g)|_x &= \left[\sqrt{\mathcal{J}_\sigma^{-1T} g \mathcal{J}_\sigma^{-1}} \mathcal{J}_\sigma \sqrt{g^{-1}} \right]_x, \\ \Lambda = \exp \omega &\longmapsto \exp \left(\frac{1}{8} \omega_{ab} [\gamma^a, \gamma^b] \right), & \mathcal{J}_\sigma(x)^\nu{}_\mu &:= \partial \sigma^\nu(x) / \partial x^\mu, \end{aligned} \quad (16)$$

and $g_{\mu\nu} := g(\partial/\partial x^\mu, \partial/\partial x^\nu)$. If the initial Dirac operator $\mathcal{D} = \not{\partial} = i\delta^\mu{}_a \gamma^a \partial/\partial x^\mu$ is the flat one, say on the (commutative) torus, then the fluctuated one can be any Dirac operator with arbitrary curvature and torsion, $\mathcal{D}_f = \not{\partial}_f = ie^{-1\mu}{}_a \gamma^a [\partial/\partial x^\mu + s(\omega_\mu)]$. We denote by s the infinitesimal version of S , e are the components of an orthonormal frame and ω is a spin connection. The local group homomorphism L is unique infinitesimally [9] and it extends the double-valued lift $SO(3) \rightarrow SU(2)$ for Pauli spinors. Note that this last double-valuedness is verified experimentally in neutron interferometry [10]

The definition of the spectral action $S_\Lambda : \mathcal{F} \rightarrow \mathbb{R}_+$ is motivated from today's definition of the unit of proper time in terms of the frequency of a particular atomic spectrum: $S_\Lambda[\mathcal{D}_f]$ is the number of eigenvalues λ of \mathcal{D}_f counted with their multiplicities such that $|\lambda| \leq \Lambda$. To compute the asymptotic behaviour of the spectral action for large eigenvalues, $\Lambda \rightarrow \infty$, we introduce a regulator, a differentiable function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of sufficiently fast decrease and put

$$S_\Lambda^h[\mathcal{D}_f] := \text{tr} [h(\mathcal{D}_f^2/\Lambda^2)]. \quad (17)$$

If we allowed discontinuous regulators, the characteristic function of the unit interval for h would reproduce S_Λ . With differentiable regulators, we obtain for 4-dimensional Riemannian manifolds M without torsion the following asymptotic expansion:

$$S_\Lambda = \int_M \left[\frac{1}{16\pi G} (2\Lambda_c - R) + a(5R^2 - 8\text{Ricci}^2 - 7\text{Riemann}^2) \right] dV + O(\Lambda^{-2}), \quad (18)$$

where the cosmological constant is $\Lambda_c = (6h_0/h_2)\Lambda^2$, Newton's constant is $G = (3\pi/h_2)\Lambda^{-2}$ and $a = h_4/(5760\pi^2)$. The spectral action is universal in the sense that the regulator h only enters through its first three 'moments', $h_0 := \int_0^\infty u h(u) du$, $h_2 := \int_0^\infty h(u) du$ and $h_4 = h(0)$. For small curvature, the spectral action therefore reproduces the Einstein-Hilbert action with a positive cosmological constant. In Robertson-Walker timespaces the curvature square term in the spectral action vanishes identically, because up to the Euler characteristic, it is proportional to the square of the Weyl tensor $C_{\mu\nu\rho\sigma}$.

Now let us look at the fluctuating Dirac operators for 4-dimensional almost commutative spaces and their spectral actions. We will find in addition to the above gravity action an entire Yang-Mills-Higgs action including the Higgs potential that produces spontaneous symmetry breaking.

Our first task is to lift the automorphisms of the algebra \mathcal{A} in the 0-dimensional triple to the Hilbert space. The most general such algebra is a direct sum of matrix algebras with real, complex or quaternionic entries, $\mathcal{A} = M_n(\mathbb{R})$, $M_n(\mathbb{C})$, or $M_n(\mathbb{H})$. All their

automorphisms g in the connected component of the identity are inner, for all $a \in \mathcal{A}$, $g(a) = uau^{-1}$ for some unitary $u \in U(\mathcal{A})$. This unitary is however ambiguous by central unitaries and we have for the connected component of the identity in the automorphism group $\text{Aut}(\mathcal{A})^e = O(n)/\mathbb{Z}_2$, $U(n)/U(1) = SU(n)/\mathbb{Z}_n$ or $USp(n)/\mathbb{Z}_2$. Immediately, the following lift comes to mind, $L(g) = \rho(g)J\rho(g)J^{-1}$. It is at most double-valued for real and quaternionic entries but it can have a continuous infinity of values for complex entries. Therefore we centrally extend it [11]. Let us illustrate the central extension for the noncommutative 0-dimensional example Eqs. (6–11), $\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$. We have

$$\text{Aut}(\mathcal{A})^e = \text{In}(\mathcal{A}) = SU(2)/\mathbb{Z}_2 \times U(3)/U(1) \leftarrow SU(2)/\mathbb{Z}_2 \times U(3) \ni (v, w), \quad (19)$$

where the central unitaries in $U(1)$ are parameterized by $\det w$ and we consider the following central extensions of the lift,

$$L(v, w) := \rho(v, (\det w)^{q_1}, (\det w)^{q_2} w) J \rho(v, (\det w)^{q_1}, (\det w)^{q_2} w) J^{-1}, \quad (20)$$

with integer or rational exponents q_1, q_2 . For concreteness we put $q_1 = 1$, $q_2 = 0$.

Now let us fluctuate the Dirac operator $\mathcal{D}_t = \not{\partial} \otimes 1 + \gamma_5 \otimes \mathcal{D}$ with both timespace diffeomorphisms σ and “internal” automorphisms g where $\not{\partial}$ is again the flat Dirac operator on the torus. Note that in almost commutative triples the internal automorphisms become timespace dependent after tensorisation, g is a gauge transformation. After some computation, we obtain the fluctuated Dirac operator of the form $\mathcal{D}_{t_f} = \not{\partial}_f \otimes 1 + \gamma_5 \otimes \mathcal{D}_f$ with

$$\not{\partial}_f = ie^{-1\mu}{}_a \gamma^a [\partial/\partial x^\mu + s(\omega_\mu) + \ell(A_\mu)], \quad A_\mu(x) \in su(2) \times u(1) \times su(3). \quad (21)$$

We denote by ℓ the infinitesimal version of the group homomorphism L in Eq. (20). The timespace Dirac operator becomes covariant with respect to a Yang–Mills field A , that is a 1-form on the 4-dimensional commutative triple. Likewise the internal Dirac operator becomes covariant with respect to a $SU(2)$ -doublet of complex scalar fields, that is a 1-form on the 0-dimensional noncommutative triple:

$$\mathcal{M}_f = \begin{pmatrix} \begin{pmatrix} \varphi_1 m_u & -\bar{\varphi}_2 m_d \\ \varphi_2 m_u & \bar{\varphi}_1 m_d \end{pmatrix} \otimes 1_3 & 0 \\ 0 & \begin{pmatrix} -\bar{\varphi}_2 m_e \\ \bar{\varphi}_1 m_e \end{pmatrix} \end{pmatrix}. \quad (22)$$

Besides their interpretation as a parallel transport in the discrete fifth direction, the scalar fields have a second interpretation: Before the fluctuation we have two parallel 4-dimensional universes a constant distance apart. After the fluctuation, this distance becomes variable, given by the scalar fields [12].

The configuration space of almost commutative triples is parameterized by the gravitational degrees of freedom, a Riemannian metric given by an orthonormal frame e and a spin connection ω , and by a Yang–Mills connection A , a 1-form valued in the Lie algebra of the automorphism group of the matrix algebra properly lifted to the spinors and a scalar field φ valued in a unitary representation of this same group. This group, as well

as its unitary representation for φ , depends on the details of the chosen 0-dimensional triple.

Let us now proceed and compute the regulated spectral action asymptotically. For our example, with vanishing torsion and $q_2 = (q_1 - 1)/3$, we obtain:

$$\begin{aligned} S_\Lambda^h[\mathcal{D}_{tf}] = & \int_M \left[\frac{1}{16\pi G} (2\Lambda_c - R) - a C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right. \\ & + 1/(2g_2^2) \operatorname{tr} F_{\mu\nu}^{(2)*} F^{(2)\mu\nu} + 1/(4g_1^2) F_{\mu\nu}^{(1)*} F^{(1)\mu\nu} + 1/(2g_3^2) \operatorname{tr} F_{\mu\nu}^{(3)*} F^{(3)\mu\nu} \\ & \left. + \frac{1}{2} (D_\mu \varphi)^* D^\mu \varphi + \lambda |\varphi|^4 - \frac{1}{2} \mu^2 |\varphi|^2 + \frac{1}{12} |\varphi|^2 R \right] dV + O(\Lambda^{-2}). \end{aligned} \quad (23)$$

We have decomposed the Yang–Mills connection as $A = (A^{(2)}, A^{(1)}, A^{(3)}) \in su(2) \times u(1) \times su(3)$, defined the field strength $F = dA + \frac{1}{2}[A, A] = \frac{1}{2}F_{\mu\nu} dx^\mu dx^\nu$ and the covariant derivative $D\varphi = d\varphi + A^{(2)}\varphi = D_\mu \varphi dx^\mu$ with $\varphi = (\varphi_1, \varphi_2)^T$. The coupling constants are given by:

$$\Lambda_c = \frac{6h_0}{h_2} \Lambda^2, \quad G = \frac{\pi}{5h_2} \Lambda^{-2}, \quad a = \frac{3h_4}{64\pi^2}, \quad (24)$$

$$g_2^{-2} = \frac{h_4}{3\pi^2}, \quad g_1^{-2} = \frac{5}{3} \frac{h_4}{3\pi^2} q_1^2, \quad g_3^{-2} = \frac{h_4}{3\pi^2}, \quad (25)$$

$$\lambda^{-1} = \frac{h_4}{\pi^2} \frac{(3m_u^2 + 3m_d^2 + m_e^2)^2}{3m_u^4 + 3m_d^4 + m_e^4}, \quad \mu^2 = 2 \frac{h_2}{h_4} \Lambda^2. \quad (26)$$

We find indeed the entire Yang–Mills–Higgs action as a fluctuation of the gravitational one. Of course we will add to it the Dirac action, $S[\psi_t, \mathcal{D}_{tf}] = (\psi_t, \mathcal{D}_{tf}\psi_t)$, which contains the couplings of the graviton, of the Yang–Mills bosons and of the Higgs scalar to the fermions (the gravity, gauge and Yukawa couplings).

For the chosen example of the 0-dimensional, noncommutative triple and a central extension satisfying $q_2 = (q_1 - 1)/3$, we obtain the action of the standard model of electromagnetic, weak and strong forces with one generation of quarks and leptons and a massless neutrino. The addition of more generations and Dirac masses in one or more generations is straight-forward. Note however that Poincaré duality imposes a purely left-handed and therefore massless neutrino in at least one generation.

So far the spectral action has been computed for only one truly noncommutative triple, the Moyal plane, where it produces the Moyal deformation of the Yang–Mills action [13], whose quantum field theory is under active investigation *e.g.* [14].

3 Spontaneous symmetry breaking

By its very construction, the spectral action is invariant under all lifted automorphisms of its algebra. These are general coordinate transformations of commutative timespace plus gauge transformations for almost commutative spaces. Unlike the Euclidean Einstein–Hilbert action, the spectral action is positive and therefore admits ground states. They are not necessarily invariant under all lifted automorphisms, we call the stability subgroup

of a given ground state the little group. When the little group is a proper subgroup one talks about “spontaneous symmetry breaking from the big group down to the little group”. Spontaneous symmetry breaking is a key ingredient of today’s particle theory because it allows to give masses to Yang–Mills bosons and to fermions, masses that are ruled out by gauge invariance. In the following we simplify the configuration space by deleting gravity: we take timespace to be the flat 4-dimensional torus of fixed volume and look for the minima of the remainder of the spectral action (23). Putting the connection A to zero and the scalar field φ to be constant minimizes the action if the norm satisfies

$$|\overset{\circ}{\varphi}|^2 = |\overset{\circ}{\varphi}_1|^2 + |\overset{\circ}{\varphi}_2|^2 = \mu^2/(4\lambda). \quad (27)$$

A constant scalar field $\overset{\circ}{\varphi}$ with this norm is called vacuum expectation value. It breaks the gauge invariance spontaneously with little group $U(3)$ in our example. Expanding the scalar field around such a minimum, $\varphi(x) = \overset{\circ}{\varphi} + H(x)$ then induces masses for the Yang–Mills bosons outside the little group via the term $\frac{1}{2} (D_\mu \overset{\circ}{\varphi})^* D^\mu \overset{\circ}{\varphi}$, masses for the fermions via the Yukawa couplings $(\psi, \overset{\circ}{\mathcal{D}}_f \psi)$ and masses to the Higgs scalar H via the Higgs potential $\lambda|\varphi|^4 - \frac{1}{2}\mu^2|\varphi|^2$. The fermion masses are (the absolute values of) the eigenvalues of the fluctuated minimal internal Dirac operator $\overset{\circ}{\mathcal{D}}_f$. Therefore the fermion masses are not given by the parameters m_u, m_d, m_e in the initial fermionic mass matrix \mathcal{M} , Eq. (10), but from the parameters in the minimum of the fluctuated one, $\overset{\circ}{\mathcal{M}}_f$. In the standard model, they happen to be identical. This coincidence is far from being generic. In the generic case, you will start with different masses in one multiplet, say $m_u < m_d$ because this is what experiment tells you. But after fluctuating and minimizing you will get degenerate masses, $\overset{\circ}{m}_u = \overset{\circ}{m}_d$. This mass degeneracy is in conflict with quantum corrections if the subgroup defining the multiplet, here $SU(2)$, is spontaneously broken. The initial and all fluctuated internal Dirac operators have degenerate eigenvalues. Nonvanishing masses appear four times, vanishing ones twice. In the standard model, Eq. (10), we have an additional threefold degeneracy of the the quark masses m_u and m_d because of the $\otimes 1_3$. “Quarks are $SU(3)$ (colour) triplets”. All these degeneracies follow from the axioms of spectral triples: \mathcal{D} commutes with the real structure, anticommutes with chirality and the first order axiom implies the colour degeneracy. Therefore we call these kinematical degeneracies. Generically there will be additional ‘dynamical’ degeneracies following from the minimisation of the action, not so in the standard model with one generation.

The standard model has two other remarkable properties, that we will discuss now.

4 A natural selection of the standard model

A real, even spectral triple is called S^0 -**real** [2] if there is a unitary operator ϵ on \mathcal{H} , called S^0 -real structure, satisfying

$$\epsilon^2 = 1, \quad [\epsilon, \rho(a)] = 0 \text{ for all } a \in \mathcal{A}, \quad [\epsilon, \mathcal{D}] = [\epsilon, \chi] = 0, \quad \epsilon J = -J\epsilon. \quad (28)$$

The commutative triples of Riemannian spaces are never S^0 -real. For 0-dimensional triples the S^0 -reality has a physical interpretation, it excludes the existence of Majorana–Weyl

fermions, which are allowed in 0 mod 8 dimensional Euclidean space or more generally in Minkowski spaces with a difference of plus and minus signs equal to 0 mod 8.

The 0-dimensional triple of the standard model is S^0 -real,

$$\epsilon = \begin{pmatrix} 1_{15} & 0 \\ 0 & -1_{15} \end{pmatrix}. \quad (29)$$

We call a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ **reducible** if there is a proper subspace $\mathcal{H}_0 \subset \mathcal{H}$ invariant under the algebra $\rho(\mathcal{A})$ such that $(\mathcal{A}, \mathcal{H}_0, \mathcal{D}|_{\mathcal{H}_0})$ is a spectral triple. If the triple is real, S^0 -real and even, we require the subspace \mathcal{H}_0 to be also invariant under the real structure J , the S^0 -real structure ϵ and under the chirality χ such that the triple $(\mathcal{A}, \mathcal{H}_0, \mathcal{D}|_{\mathcal{H}_0})$ is again real, S^0 -real and even.

The commutative triples of Riemannian spaces are all irreducible. The triple of the standard model with one generation of quarks and leptons is also irreducible.

Let us summarize, the standard model with one generation of fermions has three properties, it is S^0 -real, irreducible and dynamically non-degenerate. It is natural to ask whether there are many other such triples. Here is a partial answer [15]:

The sum of matrix algebras, $\mathcal{A} = \bigoplus_{i=1}^N \mathcal{A}_i$ with $N = 1, 2, 3$ admits a 0-dimensional, real, S^0 -real, irreducible and dynamically non-degenerate spectral triple only if it is in the list

$N = 1$	$N = 2$	$N = 3$
void	$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathcal{C}$
		$\mathbf{2} \oplus \mathbf{1} \oplus \mathcal{C}$
		$\mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}$

Here $\mathbf{1}$ is a short hand for \mathbb{R} or \mathbb{C} and $\mathbf{2}$ for $M_2(\mathbb{R})$, $M_2(\mathbb{C})$ or \mathbb{H} .

The ‘colour’ algebra \mathcal{C} is any simple matrix algebra and has two important constraints:

- i) Its representation is “vector-like”, identical on corresponding left- and right-handed subspaces of \mathcal{H} .
- ii) The Dirac operator \mathcal{D} is invariant under $U(\mathcal{C})$, $\rho(1, 1, w) \mathcal{D} \rho(1, 1, w)^{-1} = \mathcal{D}$, for all $w \in U(\mathcal{C})$. This implies that the unitaries of \mathcal{C} do not participate in the fluctuations and are therefore unbroken, i.e. elements of the little group.

The triple with two simple algebras is the two-point space where however one point has been rendered noncommutative in order to have algebra automorphisms in the connected component of the identity.

Let us conclude this section with the Darwinean explanation of gravity: In the beginning apples fell in all directions, only those falling towards the earth survived a natural selection.

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